Mathematics (Hons.) Paper-I (Sc./Arts)

Answer any six questions.

1. (a) Define and explain the terms :
   (i) Numerically equivalent sets
   (ii) Finite sets
   (iii) Denumerable sets
   (iv) Countable sets
   (v) Uncountable sets
   Illustrate them with suitable examples.
   (b) Prove that the set Q of all rational numbers is denumerable.
2. Define cardinality of a set and cardinal numbers. Construct an arithmetic of cardinal numbers. Point out the distinction between the cardinal arithmetic and the arithmetic of the natural numbers.
3. State and prove well-ordering theorem and deduce axiom of choice from well-ordering theorem.
4. (a) A set G with an associative binary operation "." define on G, is a group iff the equations $a.x = b$ and $y.a = b$ have unique solutions in G.
   (b) Show that a group with four or fewer elements is necessarily abelian.
5. (a) Define order of an element in a group G. Prove that :
   (i) $O(a) = O(a^{-1})$
   (ii) $O(ab) = O(ba)$

   (iii) $O(a) = O(x^{-1}ax)$, for all $x \in G$, where $a, b \in G$ and $O(a)$ stands for order of a in G.
   (b) Prove that a subgroup N of a group G is a normal subgroup of G if and only if every left coset of N in G is a right coset of N in G.
6. (a) State and prove Cayley's theorem.
   (b) Prove that every cyclic group of infinite order is isomorphic to the additive group (z, +) of all integers.
7. (a) Define a Hermitian matrix and a skew-Hermitian matrix. Prove that every matrix can be expresed uniquely as a sum of a Hermitian and a skew-Hermitian matrix.
   (b) If A and B are square matrices of order n, prove that $(A + B)(A - B) = A^2 - B^2$ if $AB = BA$. Show by an example that $AB = BA$ may also be false.
8. Find the eigen vectors and eigen values of the matrix A, where

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{bmatrix}$$

9. (a) Determine the rank of matrix $A = \begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$.

   (b) Prove that a system linear non-homogeneous equation $AX = B$ is consistent iff a rank of the coefficient matrix A equals the rank of the augmented matrix [AB].
10. (a) Prove that every equation of n dimensions can have n roots and no more.
    (b) Find the condition that the roots a, b, g, d of the equation $x^4 + px^3 + qx^2 + rx + s = 0$ should be connected by the relation $\alpha\beta = \gamma\delta$.
11. (a) If $\alpha, \beta, \gamma, \delta$ be the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, find the value of $\sum \frac{\alpha\beta}{\gamma^2}$.
    (b) By a suitable transformation of variable, reduce the cubic equation $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$ to the form $z^3 + 3Hz + G = 0$ and obtain the relation between the roots of the original equation and those of the transformed equation.
12. Solve $x^3 + x^2 - 16x + 20 = 0$ by Cardon's method.