Mathematics (Hons.) Paper-III (Sc./Arts)

Answer any six questions.

1. (a) State and prove Leibnitz's theorem.
   (b) If $y^{1/m} + y^{-1/m} = 2x$, prove that $(x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0$.
2. (a) State and prove Euler's theorem on homogeneous functions of two independent variables.
   (b) If $u = x^2 \sin^{-1}\left(\frac{y}{x}\right) + e^{-y/x}$, prove that : $x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = 2\,(u - e^{-y/x})$
3. (a) State and prove Maclaurin's Series and expand cos x in powers of $x - \frac{\pi}{4}$.
   (b) Prove that $\tan \phi = r\frac{d\theta}{dr}$, where the symbols have their usual meanings. Also obtain the pedal equation of the curve $r^m = a^m \cos m\theta$.
4. (a) Find the radius of curvature for the pedal curve $p = f(r)$. Also find the radius of curvature for the curve $r = a(1 - \cos\theta)$ and show that $\frac{\rho^2}{r}$ is a constant.
   (b) Prove that the chord of curvature parallel to the axis of y for the curve $y = a \log \sec\left(\frac{x}{a}\right)$ is of constant length.

5. Evaluate any two of the following :
   (a) $\int \frac{dx}{(1 + x^2)\sqrt{(1 - x^2)}}$ (b) $\int e^x \frac{(1 + x^2)}{(1 + x)^2}\, dx$ (c) $\int (\sqrt{\tan x} + \sqrt{\cot x})\, dx$
6. (a) Evaluate $\int_0^{\pi/2} \cos^n x \cos nx\, dx$, where n is a positive integer.
   (b) Show that $\int_0^{\infty} \frac{\log(1 + x^2)}{(1 + x^2)}\, dx = \pi \log 2$
7. (a) Find the entire length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$.
   (b) Find the area of the loop of the curve $ay^2 = x^2(a - x)$.
8. (a) Find the volume of the solid generated by revolving the cardioid $r = a(1 + \cos\theta)$ about the initial line.
   (b) Prove that $n\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\, 2n$.
9. (a) State and prove Taylor's theorem with various forms of remainders.
   (b) Find the maxima and minima of the function $x^3 + y^3 - 12x - 3y + 20$.
10. (a) If a function f is continuous in a closed interval [a, b] then it possesses greatest and least values.
    (b) Prove that $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty[$.
11. (a) State and prove Cauchy general principle of Convergence.
    (b) Apply Cauchy's Principle of Convergence to prove that the sequence $\langle f_n \rangle$ defined by : $f_n = 1 + \frac{1}{4} + \frac{1}{7} + \ldots + \frac{1}{3n - 2}$ is not convergent.
12. (a) State and prove Raabe's test.
    (b) Test the convergence of the series whose nth term is $\frac{\sqrt{(n+1)} - \sqrt{(n-1)}}{n}$