Mathematics (Hons.) Paper-III (Sc./Arts)

Answer six questions.

1. (a) State $\in - \delta$ definition of limit of a function. Prove that every differentiable function is continuous.

   (b) If $y = e^{a \sin^{-1}x}$, prove that : $(1 - x^2)\, y_{n+2} - (2n + 1)\, x\, y_{n+1} - (n^2 + a^2)\, y_n = 0$

2. (a) If $f(x)$ possesses continuous derivatives of every order in the interval $(x, x + h)$, then prove that $(x + h) = f(x) + h f'(x) + \frac{h^2}{\underline{2}} f''(x) + \ldots + \frac{h^n}{\underline{n}} f^n(x) + \ldots$. *inf.*

   (b) Obtain by Maclaurin's Theorem, the first five terms in the expansion of $\log(1 + \sin x)$.

3. (a) Find the condition that the line $x \cos\alpha + y \sin\alpha = p$ may touch the curve $x^m y^n = a^{m+n}$

   (b) Prove that for the polar curve , the radius of curvature is given by :

$$p = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$$

4. (a) If $u = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$, prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u$.

   (b) If $u = f(x, y)$ is a homogeneous function of degree n, then prove that :

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 x}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n - 1)\, u$$

5. Evaluate any two of the following :

   (a) $\int \frac{dx}{\sqrt{(x - \alpha)(\beta - x)}}$ (b) $\int \frac{dx}{5 + 4 \cos x}$

   (c) $\int \frac{xe^x}{(1 + x)^2}\, dx$

6. Evaluate :

   (a) $\int_0^{\pi} \frac{dx}{a + b \cos x}$ $(a > b > 0)$ (b) $\int_0^{\frac{\pi}{2}} \log \sin x\, dx$

7. (a) Find the area of the loop of the curve $x^3 + y^3 = 3axy$.

   (b) Find the area of the loop of the curve $r^2 = a^2 \cos 2\theta$.

8. (a) Find the area of surface of a cone whose semi-vertical angle is α and base a, circle of radius r.

   (b) The cardioid $r = a(1 - \cos\theta)$ revolves about initial line. Find the volume of the figure formed.

9. (a) State and prove Lagrange's method of undetermined multipliers.

   (b) Find the minimum value of $x^2 + y^2 + z^2$ under the condition $ax + by + cz = p$.

10. (a) If $\lim_{n \to \infty} a_n = l$,, then prove that $\lim_{n \to \infty} \frac{a_1 + a_2 + \ldots + a_n}{n} = l$

    (b) If $a_n > 0$, and $\lim_{n \to \infty} a_n = l$. Show that $\lim_{n \to \infty} (a_1, a_2 \ldots a_n)^{1/n} = l$.

11. (a) State and prove Logarithmic test.

    (b) Test the convergence of the series $\frac{1}{\log 2)^p} + \frac{1}{(\log 3)^p} + \ldots + \frac{1}{\log n)^p} + \ldots$

12. (a) State and prove D'Morgan's and Bertrands Test.

    (b) Test the convergence of the series $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} x + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} x^2 + \ldots$