Mathematics (Hons.) Paper-V (Sc. & Arts)

Answer six questions, selecting at least one from each Group.

Group-A

1. (a) A necessary and sufficient condition for R-integrability of a bounded function f over I [a, b] is that for any $\in > 0$, there exists a partition P such that oscillatory sum $W(P) = U(P) - L(P) < \in$

   (b) Show that the function defined as

   $f(x) = 0$, x is rational

   $= 1$, x is irrational is not Riemann integrable in finite interval [0, 1].

2. (a) Prove that every monotonic function is R-integrable.

   (b) Prove that every continuous function is R-integrable.

3. (a) State and prove Schwartz Theorem.

   (b) Examine differentiability of the function $f(x, y) = \sqrt{|xy|}$ at the origin.

4. State and prove Implicit Function Theorem.

Group-B

5. (a) Prove that the function $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$ and $f(0) = 0$ is continuous and Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

   (b) Show that the function 1/z4, $z \neq 0$ is analytic in the given domain and determine $f'(z)$.

6. (a) Define Harmonic function. Prove that u = y3 – 3x2y is a harmonic function and find its harmonic conjugate.

   (b) If f(z) is a regular function of z, prove that :

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

7. (a) Show that bilinear transformation transforms circles into circles.

   (b) Find the image of the circle $|z - 2| = 2$ under the Mobius transformation

$$= \omega = \frac{z}{z+1}$$

8. (a) Define fixed point of a bilinear transformation T(z). Prove that every bilinear transformation $\omega = T(z)$ with exactly two finite fixed points z_1 and z_2 can be written in normal form as $\dfrac{w - z_1}{w - z_2} = K\dfrac{z - z_1}{z - z_2}$, where $K \neq 0$.

   (b) Define Gross-ratio of three distinct points z_1, z_2 and z_3. Prove that Gross-ratio is invariant under the bilinear transformation.

9. (a) Define norm on a linear space. Prove that every normed linear space E is a metric space with respect to the metric d defined by $d(x, y) = \|x - y\|$, $x, y \in E$.

   (b) In a normed linear space E, prove that $|\,\|x\| - \|y\|\,| \leq \|x - y\|$, $x, y \in E$.

10. (a) Prove that every convergent sequence (xn) of points of a metric space (E, d) is a Cauchy sequence.

    (b) Give an example of a non-convergent Cauchy sequence in a metric space.

11. (a) Prove that a metric space (x, d) is sequentially compact iff it is countably compact.

    (b) Define totally bounded metric space. Prove that a subspace M of a metric space x is totally bounded iff for any $\in > 0$ there exists a finite subset A of x such that $M \subseteq U\{S_\in(x) : \in A\}$.

12. (a) Define first countable space. Prove that any discrete topological space is first countable space. (b) Define second countable space. Prove that every second countable space is a first countable space.