Mathematics (Hons.) Paper-VI (Sc. & Arts)

Answer any six questions.

1. (a) Define automorphism on a group and prove that the mapping f defined on a group G by $f(x) = x^{-1}$ for all $x \in G$ is an automorphism of G iff G is abelian.
   (b) Prove that an infinite cyclic group has just one non-trivial automorphism.
2. (a) Define conjugacy relation on a group and prove that conjugacy relation is an equivalence relation on the group.
   (b) Let G be a finite group and $a \in G$. Prove that $O(Ca) = \frac{O(G)}{O(N(a))}$, where N(a) is the normaliser of a and Ca the conjugate class of a in G.
3. State and prove Sylow's first theorem.
4. Prove that every integral domain can be embedded in a field.
5. Define a polynomial ring R[x] over a ring R. Prove that if R is a commutative ring with unity element then so is R[x]. Further prove that if R is an integral domain then so is R[x].
6. (a) Define a Unique Factorisation Domain. If D is a Unique Factorization Domain, $a, b \in D$ and p a prime in D then prove that 'p divides a.b' implies 'p divides a' or 'p divide b'.
   (b) Prove that any two elements a and b in a Euclidean ring R have a greatest common divisor d such that $d = \lambda a + \mu b$ for some $\lambda, \mu \in R$.
7. (a) Define a vector space V over a field F. If for some $a \in F$ and $x \in V$, $ax = 0$ then prove that $a = 0$ or $x = 0$, where 0 stands for zero vector or zero scalar as appropriate.
   (b) If F is a field and $V = F^n$ consisting of all n-tuples of elements of F then show that V is a vector space over F under co-ordinate-wise linear operations.
8. (a) Define basis and dimension of a vector space. Prove that any two bases of a vector space V(F) have the same number of elements.
   (b) Define direct sum of subspaces of a vector space. If W_1 and W_2 are subspaces of a vector space V(F) prove that $V = W_1 \oplus W_2$ iff for ech x V, x can be expressed uniquely in the form $x = x_1 + x_2$ for some $x_1 \in W_1$ and $x_2 \in W_2$.
9. (a) Define a linear transformation of a vector space to another. Prove that the set L (V, V') of all linear transformation of a vector space V(F) to the V'(F) is a vector space over F under pointwise linear operations in L (V, V').
   (b) Prove that a finite dimensional vector space V (F) and its dual space V^* (F) are of the same dimensions.
10. Find theeigenvalues and eigenvectorsof the matrix $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ and prove that-theeigenvectors of A are linearly independent.
11. (a) Define an inner product space and prove that every product space is anormed linear space with srespect to the norm defined by $\| x \| = \sqrt{(x, x)}$ for each x in the space andwhere(x, x) denotes the inner product of x and x.
    (b) If $\{x_n\}$, $\{y_n\}$ are sequence in an inner product space E and $x, y \in E$ such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$ then prove that $\lim_{x \to \infty} [x_n, y_n] = [x, y]$.
12. (a) State and prove Bessel's inequality in afinite dimensional inner product space.
    (b) State and prove gram-Schmidt orthogonalisation process in a finite dimensional inner product space.