HG(3) — M (8) Func.
Anal. – Sc. & Arts

2016

Time : 3 hours

Full Marks : 70

Pass Marks : 32

Candidates are required to give their answers in their own words as far as practicable.

The questions are of equal value.

Answer any ***five*** *questions.*

1. (a) Define a normed linear space E and show that every normed linear space E is a metric space with respect to the metric d defined on E by $d(x, y) = ||x - y||$, for all $x, y \in E$.

   (b) Prove that the vector addition and scalar multiplication are continuous functions in the context of a normed linear space.

2. Let c (a, b) be the set of all real valued continuous functions on the interval (a, b). Prove that c (a, b) is a Banach space under pointwise linear operations and norm defined by $||f|| = \sup_{x \in [a, b]} |f(x)|$.

3. (a) Prove that the dual space of R^n is R^n.

   (b) Define a reflexive normed linear space and find out which of the following spaces are reflexive giving reasons for your answer :

   (i) The n-dimensional Euclidean space R^n.

   (ii) The space l_p, $1 \le p < \infty$.

   (iii) The Banach space C_o of all null sequences.

4. (a) State and prove lemma of of F. Riesz.

   (b) Let B(E) denote the set of all linear operators on a normed linear space E. Show that in the algebra B(E), multiplication is related to the norm by, $||T_1 T_2|| \le ||T_1|| \cdot ||T_2||$, for all $T_1, T_2 \in B(E)$.

5. Define dual space E^* of a normed linear space E. Prove that every normed linear space E is canonically embedded in its second dual E^{**}.

6. (a) State and prove Cauchy-Schwarz inequality in an inner product space.

   (b) Prove that inner product function is jointly continuous in any inner product space.

7. Let E be complex Banach space whose norm satisfies the parallelogram law. If an inner product is defined on E by the polarisation identity then prove that E is a Hilbert space.

8. (a) Define orthogonality of vectors in an inner product space. State and prove pythagorean theorem in an inner product space.

   (b) For any non-empty subset S of a Hilbert space H, prove that the orthogonal complement $S^{\perp}$ of S, is a closed linear subspace of H.

9. Let M be a closed linear subspace of a Hilbert space H and let $x \in H$ and $x \notin M$. Let $\epsilon = d(x, M)$. Then prove that there exists a unique vector $y_0 \in M$ such that $||x - y_0|| = \epsilon$. Moreover, this y_0 is the unique element of M for which $x - y_0$ is orthogonal to M.

10. State and prove projection theorem in a Hilbert space.