Mathematics (Hons.) Paper-V (Sc. & Arts)

Answer any six questions.

1. (a) Define Riemann integrability in two different ways and prove their equivalence.
   (b) Show that the characteristic function of the set of rationals in [0, 1] is a bounded function which is not Riemann integrable.

2. (a) If $f \in R[a,b]$ then prove that $|f| \in R[a,b]$ and $\left|\int_a^b f\right| \leq \int_a^b |f|$ is the converse true ?

(b) Show that the function f defined by $f(x) = \frac{1}{2^n}$, where $\frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}$, n = 0, 1, 2, 3, ... f (0) = 0 is R-integrable over [0, 1]. Also find its R-integral.

3. (a) State and prove Abel's test for the convergence of the integral of a product of two functions.

(b) Test the convergence of $\int_0^\infty (1-e^{-x}) \frac{\cos x}{x^2} dx$.

4. (a) State and prove Schwartz's theorem.

(b) Examine the continuity and differentiability of the function

$$f(x,y) = \frac{xy^2}{x^2+y^2};\ (x,y) \neq (0,0) \text{ and } f(0,0) = 0 \text{ at } (0,0).$$

5. (a) Define analytic function. Obtain Cauchy-Riemann equations for an analytic function.

(b) If f (z) is a regular function of z, prove that :

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2, \text{ where } z = x + iy.$$

6. (a) Define bilinear transformation. Show that the resultant of two bilinear transformation is a bilinear transformation.

(b) Show that the transformation $w = \frac{2z+3}{z-4}$ transforms the circle $x^2+y^2-4x=0$ into the straight line $4u + 3 = 0$.

7. (a) Define conformal transformation. Prove that at each point z of a domain where f (z) is analytic and $f'(z) \neq 0$, the mapping w = f (z) is conformal.

(b) Find the fixed points and normal form of the linear transformation $w = \frac{z}{z-2}$.

8. (a) Prove that $|z_1 + z_2| \geq |z_1| - |z_2|$.

(b) Prove that the area of the triangle whose vertices are the points z_1, z_2, z_3 on the Argand Diagram is $\sum\left[\frac{(z_2 - z_3)|z_1|^2}{4iz_1}\right]$

9. (a) Define a metric space. Let X be a non-empty set and d : $X \times X \to R$

defined as : $d(x,y) = \begin{cases} 0, \text{ if } x = y \\ 1, \text{ if } x \neq y \end{cases} \forall x, y \in R$ Prove that d is a

metric on X.

(b) Let (X, d) be a metric space and x, y, z be points on X, then prove that

$|d(x,y) - d(y,z)| \leq d(x,z)$.

10. (a) Let X be a metric space then prove that a subset G of X is open iff G is a union of open spheres.

(b) Prove that every metric space is a T_2-space.

11. (a) Define complete metric space. Prove that a subspace Y of a complete metric space (X, d) is complete iff Y is closed.

(b) State and prove Baire's category theorem.

12. (a) Define neighbourhoods of a point, interior and exterior points of a set and closure of a set in a topological space.

(b) If $X = \{a, b, c, d, e)$ and

$\tau\{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ be a topology on X. Find all the neighbourhoods of a, b and d in (x, τ).