Mathematics (Hons.) Paper-V (Sc. & Arts)

Answer any six questions.

1. (a) State and prove the necessary and sufficient conditions for R-integrability of a bounded function defined on the bounded interval [a, b].

   (b) Let f be the function defined on [0, 1] by

   $$f(x) = 0 \text{ when } x \text{ is irrational}$$
   $$= 1 \text{ when } x \text{ is rational}$$

   Calculate $\underline{\int_0^1} f$ and $\overline{\int_0^1} f$ hence show that $f \notin R[0,1]$.

2. (a) Let $f \in R[a,b]$ and let ϕ be a differentiable function on [a, b] such that $\phi'(x) = f(x)$ for all $x \in [a,b]$. Then $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$.

   (b) If f is continuous on [a, b], then show that f is R-integrable on [a, b].

3. (a) State comparison test for the convergence of an improper integral and hence test the convergence of the integral $\int_0^\infty \frac{\cos x}{1+x^2}\,dx$

   (b) Test the convergence of the integral $\int_0^\infty e^{-ax} \frac{\sin x}{x}\,dx,\ a \geq 0$.

4. (a) State and prove Young's theorem.

   (b) Examine the differentiability of the function $f(x) = \sqrt{|xy|}$ at the origin.

5. (a) Define modulus, amplitude and conjugate of a complex number and show that

the modulus of sum or difference of two complex number is always less than or equal to the sum of their moduli.

(b) Obtain the polar form of the Cauchy's-Riemann conditions for a function f(z) to be analytic.

6. (a) Define harmonic functions and show that, if $f(z) = u + iv$ is an analytic function, then u and v are both harmonic functions.

(b) Show that the functions $f(z) = e^{-z^{-4}}$ $(z \neq 0)$ and $f(0) = 0$ is not analytic at $z = 0$, although Cauchy-Riemann equations are satisfied at the point.

7. (a) Show the bilinear transformation transforms circles into circles.

(b) Find the bilinear transformation which transforms the point $z = 2, 1, 0$ into $w = 1, 0, i$, respectively.

8. (a) Define inverse points with respect to a circle and show that the two points z_1, z_2 are the inverse points with respect to the circle

$$z\bar{z} + b\bar{z} + \bar{b}z + c = 0$$

$$\text{if} \quad z_1 \bar{z}_2 + b\bar{z}_2 + \bar{b}z_1 + c = 0.$$

(b) Show that the function $f(z) = xy + iy$ is everywhere continuous but is not analytic.

9. (a) Define a metric space, if (E, ρ) is a metric space. then prove that (E, d) is a metric space where d is defined by :

$$d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)} \text{ for all } x, y \in E$$

(b) Prove that every open sphere is a metric space (E, d) is an open set.

10. (a) Show that every convergent sequence (x_n) of points of a metric space (E, d) is a Cauchy sequences.

(b) Give an example of a non-convergent Cauchy sequence.

11. (a) State and prove Cantor's Intersection theorem.

(b) Show that any contraction map T on a metric space (E, d) is uniformly continuous.

12. Define compactness and sequential compactness for a metric space and prove that a metric space is compact if and only if it is sequentially compact.

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