Mathematics (Hons.) Paper-V

Answer six questions, selecting at least one from each Group.

Group-A

1. (a) Define Riemann integrability in two different ways and prove their equivalence.

   (b) Give an example of a function f which is not integrable but | f | in integrable.

2. (a) Prove that if a function f (x) iscontinuous in [a, b], then it is R-integrable over [a, b].

   (b) Evaluate the R-integral $\int_0^{?} f(x)\,dx$, where $f(x) = \frac{1}{2^{n+1}}$ for $\frac{1}{2^{n+1}} < x < \frac{1}{2^n}$

   (n = 0, 1, 2, ...)

3. (a) State and prove Abel's test for the convergence of the integral of a product of two functions.

   (b) Test for convergence $\int_0^{\infty} e^{-x} \sin \frac{1}{x}\,dx$.

4. (a) If f_x and f_y exist in a neighbourhood of the point (a, b) and f_x and f_y differentiable at (a, b), then prove that $f_{xy} = f_{yx}$ at (a, b).

   (b) If $f(x, y) = \frac{x^2 y^2}{x^2 y^2 + (x - y)^2}$, then discuss the existence of the repeated and double limits at (0, 0).

Group-B

5. (a) Define continuity and differentiability of a function of complex variable.

   (b) Prove that the function $|z|^2$ is continuous everywhere but nowwhere differentiable except at the origin.

6. If a function $f(z) = u(x, y) + iv(x, y)$ is differentiable at any point $z = x + iy$, then the partial derivatives u_x, u_y, v_x, v_y should exist and satisfy the equations $u_x = v_y$ and $u_y = -v_x$.

7. (a) Show that real and imaginary parts of an analytic function satisfy Laplace's equation.

   (b) Show that the function $u = x^3 - 3xy^2$ is harmonic and find the corresponding analytic function.

8. Define conformal mapping. State and prove sufficient conditions of $W = f(z)$ to represent a conformal mapping.

9. (a) Define a metric space. If (E, ρ) is a metric space, then provethat (E, d) is a metric space where d is defined by $d(x, y) = \frac{\rho(x, y)}{1 + \rho(x, y)}$, $x, y \in E$.

   (b) Prove that an open sphere in a metric space is an open set.

10. (a) If (Ed, d) is a metric space, then prove that for each pair x, y of distinct points of E, $\exists$ a neighbourhood U of x and a neighbourhood V of y such that $U \cap V = \phi$.

    (b) Prove that a subset F of a metric space (E, d) isclosed iff $F = \overline{F}$.

11. State and prove Baire's category theorem.

12. Prove that metric space (C, d) of complex plane, where $d(z_1, z_2) = |z_1 - z_2|$ $z_1, z_2 \in C$, is complete.