Mathematics (Hons.) Paper-VI (Sc./Arts)

Answer six questions, selecting at leat one from each Group.

Group-A

1. (a) Prove that the set of all the automorphisms of a group with resultant composition is a group.

   (b) Define inner automorphism of a group. Prove that the set of all inner automorphisms of a group G is a subgroup of the group of its automorphisms.

2. (a) Show that centre of a group G is a subgroup of G.

   (b) Prove that the relation of conjugacy is an equivalence relation in a group G.

3. State and prove Sylow's First Theorem.

Group-B

4. (a) Prove that every field is a principal ideal ring.

   (b) Give an example of an ideal which is prime ideal and also a maximal ideal.

5. Construct the polynomial ring R[x] over the arbitrary ring R by defining suitable composition of addition and multiplication over the set of polynomials over R.

6. (a) Define unique factorization domain. If a, b are two arbitrary elements of a unique factorization domain D and p is a prime element of D, then

   $p/ab \Rightarrow$ either p/a or p/b

   (b) A principal ideal domain is unique factorization domain.

Group-C

7. (a) Define vector space with an example.

   (b) The necessary and sufficient condition for a non-empty subset M of a linear space V(F) to be a subspace of V is $\alpha, \beta \in F, x, y \in M \Rightarrow \alpha x + \beta y \in M$.

8. (a) Show that the mapping $T : R_2 \rightarrow R_3$ defined as $T(x, y) = (x + y, x - y, y)$ is a linear transformation. Find the rank and nullity of T.

   (b) The dual space V^* of a finite dimensional vector space V (F) of dimension n is itself of dimension n. Prove this.

9. (a) Define eigen values and eigen vectors of a linear transformation , where V is a vector space over a scalar field.

   (b) Let $A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ be a given matrix. Find a matrix P such that P-1 is a diagonal matrix.

10. State and prove Bessel's inequality for finite dimensional space.

11. (a) Define inner product space and prove that every inner product space is normed linear space with respect to the norm given by $\| x \| = \sqrt{x/x}$

    (b) Let $x_n \rightarrow x$ and $y_n \rightarrow y$ in an inner product space E, then prove that

    $$(x_n, y_n) \rightarrow (x, y).$$

12. If $\{e_1, e_2, ... e_n\}$ be a finite orthonormal set in a Hilbert space H and x be any vector in H, then prove that :

    (i) $\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$ (ii) $x - \sum_{i=1}^{n} (x, e_i)\, e_i \perp e_j$ for each j.