Mathematics (Hons.) Paper-VIII (Fun.Anal.)

Answer any five questions.

1. (a) Define 'norm' on a linear space E. Prove that every normed linear space E is a metric space with respect to the metric d defined by $d(x, y) = \| x - y \|$ for all x, y $\in$ E.

   (b) Prove that vector addition is a continuous function in the context of normed linear space.

2. Prove that the set Cn of all n-tuples z = (z1, z2, ..., zn) of complex numbers is a complex Branch space if we define for

   $z = (z_1, z_2, \ldots\ldots, z_n)$, $w = (w_1, w_2, \ldots, w_n) \in C^n$ and $\lambda \in C$,

   $(z_1, z_2, \ldots\ldots, z_n) + (w_1, w_2, \ldots, w_n) = (z_1 + w_1, z_2 + w_2, \ldots, z_n + w_n)$

   $\lambda (z_1, z_2, \ldots\ldots, z_n) = (\lambda z_1, \lambda z_2, \ldots\ldots, \lambda z_n$ and $\| z \| = \left(\sum_{i=1}^{n} |z_i|^2 \right)^{1/2}$ as norm of z.

3. (a) State and prove Minkowski's Inequality.

   (b) Construct a metric space which is not a normed linear space.

4. (a) A linear transformation T from a normed linear space E into a normed linear space F is continuous iff it is continuous at the origin.

   (b) A linear transformation T from a normed linear space E into a normed linear space F is continuous iff T is bounded in the sense that there exists a positive real number m such that : $\| T(x) \| \leq m \| x \|$ for all $x \in E$.

5. Prove that dual space of every normed linear space is a Branch space.

6. (a) State and prove the Lemma of F. Ries z.

   (b) Prove that conjugate space of l_p is l_q.

7. (a) Define Hilbert Space R^n. Prove that real linear space Rn is a Hilbert space with respect to inner product defined by $(x, y) = \sum_{i=1}^{n} x_i y_i$ for

   $(x_1, x_2, \ldots\ldots, x_n)$, $y\,(y_1, y_2, \ldots, y_n)$ of Rn.

   (b) Prove that inner product function is jointly continuous.

8. (a) State and prove Cauchy-Schwartz inequality in a Hilbert space.

   (b) State and prove Polarisation identity in a Hilbert space.

9. (a) Prove that a sphere in a normed linear space is a convex set.

   (b) Prove that intersection of any family of convex sets is a convex set but union of two convex sets may not be a convex set.

10. State and prove Projection theorem in a Hilbert space.

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